Particle mechanics and relativity.

Binomial Theorem

By the Binomial Theorem,
$(1+\varepsilon)^p = 1 + \frac{p\varepsilon}{1!} + \frac{p(p-1)\varepsilon^2}{2!} + \frac{p(p-1)(p-2)\varepsilon^3}{3!} + \cdots$
For very small ε, $(1+\varepsilon)^p \approx 1 + \frac{p\varepsilon}{1!}$. Thus
$\sqrt{1-v^2} = \left(1-v^2\right)^{\frac{1}{2}} \approx 1 + \left(\frac{1}{2}\right)\left(-v^2\right) = 1 - \frac{v^2}{2}$, and
$\left(1-v^2\right)^{-\frac{1}{2}} \approx 1 + \left(-\frac{1}{2}\right)\left(-v^2\right) = 1 + \frac{v^2}{2}$.

Trajectory of a particle is time-like

Since $(\triangle\tau)^2 = (\triangle t)^2 - (\triangle x)^2$.
The proper time interval, $\triangle\tau$, is positive if $\triangle t > \triangle x$ (it has more time in it than space), and the interval is called time-like. In this case, we can always find a Lorentz frame of reference where the two points have the same point in space $x' = 0$.

The proper time interval, $\triangle\tau$, is positive if $\triangle x > \triangle t$ (it has more space in it than time), and the interval is called space-like. In this case, we can always find a Lorentz frame of reference where the two points have the same time $t' = 0$ or occur earlier than each other (the relativity of simultaneity).

Since particles can't move faster than c, particles move in time-like fashion.

Four vectors

$\triangle x^\mu = (\triangle t, \triangle \overrightarrow{x}) = \left(\triangle t, \triangle x^i\right)$ where $i = 1, 2, 3$.
Since $(\triangle\tau)^2 = (\triangle t)^2 - (\triangle x)^2$, $\triangle\tau = \sqrt{(\triangle t)^2 - \left(\frac{\triangle x}{c}\right)^2}$, and is invariant for all observers.

Four velocity $U^\mu = \frac{dX^\mu}{d\tau}$ where $\mu = 0, 1, 2, 3$.
$\overrightarrow{v} = \frac{d\overrightarrow{x}}{dt}$ or $v^i = \frac{dx^i}{dt} = \frac{dx^i}{d\tau}\frac{d\tau}{dt} = U^i\frac{d\tau}{dt}$.
$U^0 = \frac{dt}{d\tau} \Longrightarrow \frac{d\tau}{dt} = \frac{1}{U^0}$.
Since $\triangle\tau = \sqrt{(\triangle t)^2 - \left(\frac{\triangle x}{c}\right)^2}$,
then $d\tau = \sqrt{dt^2 - (d\overrightarrow{x})^2} = \sqrt{1 - \frac{\left(d\overrightarrow{x}\right)^2}{dt^2}}dt = \sqrt{1-v^2}dt$.
Thus $\frac{d\tau}{dt} = \sqrt{1-v^2}$ and $\frac{dt}{d\tau} = U^0 = \frac{1}{\sqrt{1-v^2}}$.
So $v^i = U^i\frac{d\tau}{dt} = U^i\sqrt{1-v^2}$, thus $U^i = \frac{v^i}{\sqrt{1-v^2}}$. Thus the 4 velocity vector has only 3 independent components.
Key equalities: $U^0 = \frac{1}{\sqrt{1-v^2}}$, and $U^i = \frac{v^i}{\sqrt{1-v^2}}$.

Likewise $(\triangle\tau)^2 = (\triangle t)^2 - (\triangle x)^2$, $\frac{(\triangle\tau)^2}{(\triangle\tau)^2} = \frac{(\triangle t)^2}{(\triangle\tau)^2} - \frac{(\triangle x)^2}{(\triangle\tau)^2}$, so
$1 = \frac{(\triangle t)^2}{(\triangle\tau)^2} - \frac{(\triangle x)^2}{(\triangle\tau)^2} = \left(U^0\right)^2 - \left(U^i\right)^2 = \frac{1}{1-v^2} - \frac{\left(v^i\right)^2}{1-v^2} = 1$.

Laws governing the motion of a free particle; the principle of least action.

A particle will follow a trajectory that minimizes the action and the action will be the same invarient in every inertial frame.

Action equals the integral over the trajectory, and τ along the trajectory is invariant.
Action $A = \sum\limits_a^b \triangle\tau$ has to be minimized holding the end points fixed.
We can multiplying A by $-m$,
$A = -m\sum\limits_a^b \triangle\tau = -m\int d\tau = -m\int\sqrt{dt^2 - (d\overrightarrow{x})^2} = -m\int\sqrt{1 - \frac{(d\overrightarrow{x})^2}{dt^2}}dt$
$A = -m\int\sqrt{1 - \left(\frac{dx^i}{dt}\right)^2}dt = -m\int\sqrt{1 - \left(\dot{x^i}\right)^2}dt.$

In the ordinary principle of least action is the integral of the Lagrangian, $A = -m\int L(\dot{x}, x)\,dt$. When no forces are present, L only depends on velocity so $A = -m\int L(\dot{x})\,dt$.
Thus $L = -m\sqrt{1 - \left(\dot{x^i}\right)^2} = -m\sqrt{1 - (\dot{x})^2 - (\dot{y})^2 - (\dot{z})^2} = -m\sqrt{1-v^2}$ encapsulates motion of a free particle in a way which is independent of the inertial frame of reference.

When v is small, relative to the spped of light c, using the Binomial Theorem, $L = -m\sqrt{1-v^2} = m\left(1 - \frac{v^2}{2}\right) = -m + \frac{mv^2}{2}$.

Conservation of momentum and energy.

Momentum is conserved because of translation invariance, i.e., momentum only depends on velocity, not position. In the Lagrangian point of view,
$p_x = \frac{\partial L}{\partial \dot{x}} = \frac{-m(-2\dot{x})}{2\sqrt{1-v^2}} = \frac{mv_x}{\sqrt{1-v^2}} = mU^x$,
$p_y = \frac{\partial L}{\partial \dot{y}} = \frac{-m(-2\dot{y})}{2\sqrt{1-v^2}} = \frac{mv_y}{\sqrt{1-v^2}} = mU^y$,
$p_z = \frac{\partial L}{\partial \dot{z}} = \frac{-m(-2\dot{z})}{2\sqrt{1-v^2}} = \frac{mv_z}{\sqrt{1-v^2}} = mU^z$.

The four vector of displacement, i.e., $\left(\triangle t, \triangle x^i\right)$ is simply an interval.
If $\triangle x = 0$, $\triangle x$ changes under a Lorentz Transform (because it picks up some $\triangle t$), so it is not invariant.
If $\triangle x = 0$, and $\triangle t = 0$, the Four vector is zero - which is an invariant statement. If only $U^i = 0$, it is not invariant in a non-stationary frame of reference.

Momentum Conservation.
$\overrightarrow{p_{initial}} = \overrightarrow{p_{final}} \Longrightarrow \overrightarrow{p_{initial}} - \overrightarrow{p_{final}} = 0$ is the invariant vector equation, if only there is a fourth component that turns momentum into a 4-vector which is conserved $mU^0 = p_0$.

The Hamiltonian, H, is used to calculate energy.
H is given in terms of the Lagrangian:
$p_i = \frac{m\dot{x}^i}{\sqrt{1-v^2}}$, $L = -m\sqrt{1-v^2}$
$H = \sum\limits_i \dot{x}^i p_i - L = \sum\limits_i \dot{x}^i mU^i + m\sqrt{1-v^2} = \sum\limits_i \dot{x}^i \frac{mv^i}{\sqrt{1-v^2}} + m\sqrt{1-v^2}$
$= \sum\limits_i \frac{m(\dot{x})^2}{\sqrt{1-v^2}} + m\sqrt{1-v^2} = \frac{mv^2}{\sqrt{1-v^2}} + m\sqrt{1-v^2}$
$= \frac{mv^2}{\sqrt{1-v^2}} + \frac{m\left(1-v^2\right)}{\sqrt{1-v^2}} = \frac{m}{\sqrt{1-v^2}}$ so $p_0 = \frac{m}{\sqrt{1-v^2}} = mU^0$.

Conservation of momentum becomes the conservation of 4 momentum for a truely isolated system.
$mU^0 = p_0 = \frac{m}{\sqrt{1-v^2}}$.

By the binomial theorem, $\left(1-v^2\right)^{-\frac{1}{2}} \approx 1 + \left(-\frac{1}{2}\right)\left(-v^2\right) = 1 + \frac{v^2}{2}$, thus $p_0 = \frac{m}{\sqrt{1-v^2}} = m + \frac{mv^2}{2}$ where $\frac{mv^2}{2}$ is ordinary Newtonia kinetic energy.
In order to make the equation dimensionally consistent, we multiply by c^2.
$E = mc^2 + \frac{mv^2}{2}$.

Massless particles think of energy as a function of momentum instead of velocity.
In Newtonian mechanics, $E = \frac{mv^2}{2} = \frac{p^2}{2m}$.
In Relativistic mechanics, $\left(U^0\right)^2 - \left(U^x\right)^2 - \left(U^y\right)^2 - \left(U^z\right)^2 = 1$ (from earlier), so $\left[m^2\left(U^0\right)^2 - \left(U^x\right)^2 - \left(U^y\right)^2 - \left(U^z\right)^2\right] = E^2 - p^2 = m^2$.
so $E = \sqrt{m^2 + p^2} = \sqrt{m^2c^4 + p^2c^2}$.
Putting back in the speed of light, $E = \sqrt{m^2c^4 + p^2c^2}$
Then $\lim\limits_{m \to 0} \sqrt{m^2c^4 + p^2c^2} = c \mid p \mid$. This is true for photons, neutrinos and gravitons, but not for particles moving slower than c.

Positronium.

Positronium, Ps, is a neutrally charged electron and positron particle in orbit around each other. It has slightly less mass than 2 electrons (less because of binding forces); it will decay into two photons (electro-magnetic energy mass is not preserved as the photons are massless, but energy and momentum are conserved.
If Positronium is at rest, its momentum is 0. The photons will go off with opposite and equal momentum so that the final momentum is 0.

In terms of energy, the energy for at rest Positroium is mc^2. So the energy of 2 photons is $2c \mid p \mid = mc^2$ in order for total energy to be conserved. Thus $\mid p \mid = \frac{mc}{2}$ is the momentum of the photon.