Exercise 1.2.4, page 54
from *Introduction to Tensor Calculus* by J.H. Heinbockel

Consider the Lorentz transformation from relativity theory having the velocity parameter V, c is the speed of light and $x^4 = t$ is time.

$$\begin{pmatrix} \bar{x}^1 \\ \bar{x}^2 \\ \bar{x}^3 \\ \bar{x}^4 \end{pmatrix} = \mathrm{T}_V \begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix}, \text{ where } \mathrm{T}_V \begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix} = \begin{pmatrix} \frac{x^1 - Vx^4}{\sqrt{1-\frac{V^2}{c^2}}} \\ x^2 \\ x^3 \\ \frac{x^4 - \frac{Vx^1}{c^2}}{\sqrt{1-\frac{V^2}{c^2}}} \end{pmatrix}. \text{ If we let}$$

$$\mathrm{T}_V = \begin{pmatrix} \left(1-\frac{V^2}{c^2}\right)^{-\frac{1}{2}} & 0 & 0 & -V\left(1-\frac{V^2}{c^2}\right)^{-\frac{1}{2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{V\left(1-\frac{V^2}{c^2}\right)^{-\frac{1}{2}}}{c^2} & 0 & 0 & \left(1-\frac{V^2}{c^2}\right)^{-\frac{1}{2}} \end{pmatrix}, \text{ then} \mathrm{T}_V \begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix} = \begin{pmatrix} \frac{x^1 - Vx^4}{\sqrt{1-\frac{V^2}{c^2}}} \\ x^2 \\ x^3 \\ \frac{x^4 - \frac{Vx^1}{c^2}}{\sqrt{1-\frac{V^2}{c^2}}} \end{pmatrix}.$$

Show this set of transformations is a group, by establishing:

(i) $V = 0$ gives the identity transformation, T_0.

When $V = 0$, then $\mathrm{T}_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, the identity.

(ii) $\mathrm{T}_{V_2} \cdot \mathrm{T}_{V_1} = \mathrm{T}_{V_3}$ requires that $V_3 = \frac{V_1+V_2}{1+\frac{V_1 V_2}{c^2}}$.

If we let $\beta_v = \frac{v}{c}$, and let $\gamma_v = \left(1-\beta_v^2\right)^{-\frac{1}{2}}$, then $\mathrm{T}_v = \begin{pmatrix} \gamma_v & 0 & 0 & -\beta_v\gamma_v c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{\beta_v\gamma_v}{c} & 0 & 0 & \gamma_v \end{pmatrix}$.

It follows that

$$\mathrm{T}_{v_3} = \begin{pmatrix} \gamma_{v_3} & 0 & 0 & -\beta_{v_3}\gamma_{v_3} c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{\beta_{v_3}\gamma_{v_3}}{c} & 0 & 0 & \gamma_{v_3} \end{pmatrix}, \text{ but}$$

$$\mathrm{T}_{v_3} = \mathrm{T}_{v_2}\mathrm{T}_{v_1} = \begin{pmatrix} \gamma_{v_2} & 0 & 0 & -\beta_{v_2}\gamma_{v_2} c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{\beta_{v_2}\gamma_{v_2}}{c} & 0 & 0 & \gamma_{v_2} \end{pmatrix} \begin{pmatrix} \gamma_{v_1} & 0 & 0 & -\beta_{v_1}\gamma_{v_1} c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{\beta_{v_1}\gamma_{v_1}}{c} & 0 & 0 & \gamma_{v_1} \end{pmatrix}.$$

$$= \begin{pmatrix} \gamma_{v_2}\gamma_{v_1} + \beta_{v_2}\gamma_{v_2} c\frac{\beta_{v_1}\gamma_{v_1}}{c} & 0 & 0 & -\gamma_{v_2}\beta_{v_1}\gamma_{v_1} c - \gamma_{v_1}\beta_{v_2}\gamma_{v_2} c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{\gamma_{V_1}\beta_{V_2}\gamma_{V_2}}{c} - \frac{\gamma_{V_2}\beta_{V_1}\gamma_{V_1}}{c} & 0 & 0 & \beta_{V_1}\gamma_{V_1} c\frac{\beta_{V_2}\gamma_{V_2}}{c} + \gamma_{V_2}\gamma_{V_1} \end{pmatrix}$$

$$= \begin{pmatrix} \gamma_{v_1}\gamma_{v_2} + \gamma_{v_1}\gamma_{v_2}\beta_{v_1}\beta_{v_2} & 0 & 0 & -c\left(\gamma_{v_1}\gamma_{v_2}\beta_{v_1} + \gamma_{v_1}\gamma_{v_2}\beta_{v_2}\right) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{\left(\gamma_{v_1}\gamma_{v_2}\beta_{v_1} + \gamma_{v_1}\gamma_{v_2}\beta_{v_2}\right)}{c} & 0 & 0 & \gamma_{v_1}\gamma_{v_2} + \gamma_{v_1}\gamma_{v_2}\beta_{v_1}\beta_{v_2} \end{pmatrix}$$

$$= \begin{pmatrix} \gamma_{v_1}\gamma_{v_2}\left(1 + \beta_{v_1}\beta_{v_2}\right) & 0 & 0 & -c\gamma_{v_1}\gamma_{v_2}\left(\beta_{v_1} + \beta_{v_2}\right) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{\gamma_{v_1}\gamma_{v_2}\left(\beta_{v_1} + \beta_{v_2}\right)}{c} & 0 & 0 & \gamma_{v_1}\gamma_{v_2}\left(1 + \beta_{v_1}\beta_{v_2}\right) \end{pmatrix}.$$

So

$$\begin{aligned}
\gamma_{v_3} = \gamma_{v_1}\gamma_{v_2}\left(1 + \beta_{v_1}\beta_{v_2}\right) &= \left(1 - \frac{v_1^2}{c^2}\right)^{-\frac{1}{2}} \left(1 - \frac{v_2^2}{c^2}\right)^{-\frac{1}{2}} \left(1 + \frac{v_1 v_2}{c^2}\right) \\
&= \left(1 - \frac{v_1^2}{c^2}\right)^{-\frac{1}{2}} \left(1 - \frac{v_2^2}{c^2}\right)^{-\frac{1}{2}} \left(\left(1 + \frac{v_1 v_2}{c^2}\right)^2\right)^{\frac{1}{2}} \\
&= \left(1 - \frac{v_1^2}{c^2}\right)^{-\frac{1}{2}} \left(1 - \frac{v_2^2}{c^2}\right)^{-\frac{1}{2}} \left(\left(\frac{1}{1+\frac{v_1 v_2}{c^2}}\right)^2\right)^{\frac{1}{-2}} \\
&= \left(\frac{\left(1-\frac{v_1^2}{c^2}\right)\left(1-\frac{v_2^2}{c^2}\right)}{\left(1+\frac{v_1 v_2}{c^2}\right)^2}\right)^{-\frac{1}{2}} \\
&= \left(\frac{1-\frac{v_1^2}{c^2}-\frac{v_2^2}{c^2}+\frac{v_1^2 v_2^2}{c^4}}{\left(1+\frac{v_1 v_2}{c^2}\right)^2}\right)^{-\frac{1}{2}} = \left(\frac{1+\frac{2v_1 v_2}{c^2}+\frac{v_1^2 v_2^2}{c^4}-\frac{1}{c^2}\left(v_1^2+2v_1 v_2+v_2^2\right)}{\left(1+\frac{v_1 v_2}{c^2}\right)^2}\right)^{-\frac{1}{2}} \\
&= \left(\frac{\left(1+\frac{v_1 v_2}{c^2}\right)^2-\frac{1}{c^2}(v_1+v_2)^2}{\left(1+\frac{v_1 v_2}{c^2}\right)^2}\right)^{-\frac{1}{2}} = \left(1 - \frac{\frac{(v_1+v_2)^2}{c^2}}{\left(1+\frac{v_1 v_2}{c^2}\right)^2}\right)^{-\frac{1}{2}}
\end{aligned}$$

So $\gamma_{v_3} = \left(1 - \beta_{v_3}^2\right)^{-\frac{1}{2}} = \left(1 - \frac{\frac{(v_1+v_2)^2}{c^2}}{\left(1+\frac{v_1 v_2}{c^2}\right)^2}\right)^{-\frac{1}{2}} = \left(1 - \left(\frac{\frac{v_1+v_2}{c}}{1+\frac{v_1 v_2}{c^2}}\right)^2\right)^{-\frac{1}{2}}$.

Thus $\beta_{v_3} = \frac{v_3}{c} = \frac{\frac{v_1+v_2}{c}}{1+\frac{v_1 v_2}{c^2}}$, so $v_3 = \frac{v_1+v_2}{1+\frac{v_1 v_2}{c^2}}$.

(iii) $\mathrm{T}_{V_2} \cdot \mathrm{T}_{V_1} = \mathrm{T}_0$ requires that $V_2 = -V_1$.
If $V_2 = -V_1$, then $\beta_{v_3} = \frac{\frac{v_1+v_2}{c}}{1+\frac{v_1 v_2}{c^2}} = \frac{\frac{v_1-v_1}{c}}{1-\frac{v_1 v_1}{c^2}} = 0$, and $\gamma_{v_3} = \left(1 - \beta_{v_3}^2\right)^{-\frac{1}{2}} = 1$.

Thus $\mathrm{T}_{v_3} = \begin{pmatrix} \gamma_{v_3} & 0 & 0 & -\beta_{v_3}\gamma_{v_3} c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{\beta_{v_3}\gamma_{v_3}}{c} & 0 & 0 & \gamma_{v_3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathrm{T}_0$.