Derivation of the Lorentz Transformation.

In Newtonian Relativity

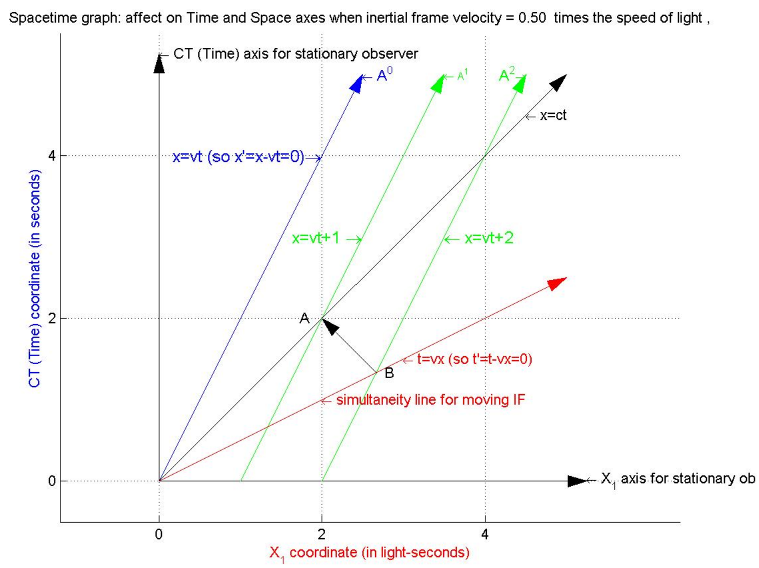
Let $x = ct$ equal the distance x travelled by light in time t. Similarly, let vt equal the distance travelled by an observer moving at velocity v in time t.

Now if we let x' be the the position of of the moving observer in a second inertial reference frame, and t' be the time in the moving observer's inertial frame (IF) of reference, then $x' = x - vt$ and $t' = t$ (since synchronized clocks have the same time - i.e., sumultaneity is the same in all inertial frames (IFs).

Along a light ray, since $x = ct$, then $x' = x - vt = ct - vt = (c - v)\,t$.
Since $t' = t$, it is also true that $x' = (c - v)\,t'$.
Thus the light ray moves at speed $c - v$ in the moving inertial reference frame (IF).

In Special Relativity (SR)

Assume velocity v is measured in terms of c, the speed of light.
We postulate that the speed of light is c, a constant, in every inertial frame (IF).



At Point A.
Assume $c = 1$, then $x = ct \Longrightarrow x_A = t_A$,
and observe that A is on the line where $x_A = vt_A + 1$.
Thus $x_A = vt_A + 1 \Longrightarrow t_A = vt_A + 1 \Longrightarrow t_A - vt_A = 1$. So $t_A = \frac{1}{1-v}$.
Then since $x_A = t_A$, then it is also true that $x_A = \frac{1}{1-v}$.
Thus at Point A, $t_A = \frac{1}{1-v}$ and $x_A = \frac{1}{1-v}$.

On the line AB.
The line AB has a slope of $-c$, or -1. Thus $t = -x + constant$, or $x + t = constant$.
Since $t = \frac{1}{1-v}$ and $x = \frac{1}{1-v}$. then $x + t = \frac{1}{1-v} + \frac{1}{1-v} = \frac{2}{1-v}$.
In particular, all along line AB, $x_B + t_B = \frac{2}{1-v}$.

At Point B.
Observe point B is the intersection of line AB and the line where $x_B = vt_B + 2$.
Thus $x_B = vt_B + 2 \Longrightarrow x_B - vt_B = 2$ and $x_B + t_B = \frac{2}{1-v}$.
Subtracting the first equation from the second, we get
$t_B + vt_B = \frac{2}{1-v} - 2$. Thus
$t_B\,(1+v) = \frac{2}{1-v} - 2 = \frac{2v}{1-v}$, so $t_B = \frac{2v}{1-v^2}$.

Then since $x_B + t_B = \frac{2}{1-v} \Longrightarrow x_B = \frac{2}{1-v} - t_B$, then $x_B = \frac{2}{1-v} - \frac{2v}{1-v^2} = \frac{2}{1-v^2}$.
Thus at Point B, $t_B = \frac{2v}{1-v^2}$ and $x_B = \frac{2}{1-v^2}$.

The slope of the line of simultaneity is $\frac{t_B}{x_B} = \frac{\frac{2v}{1-v^2}}{\frac{2}{1-v^2}} = v$, so $t_B = vx_B$,
or more generally $t = vx$ is the equation for the line of simultaneity where $t' = 0$.

If $c = c$ in every inertial reference frame (IF), then what is synchronous in one IF is not always synchronous in another IF. So what is the relationship between x and t and x' and t'?

At this juncture, we only know that $x' = 0$ when $x = vt$ and $t' = 0$ when $t = vx$. So assume that when $x' \neq 0$ that $x' = (x - vt)\, f(v)$ where $f(v)$ is a function of the magnitude of the velocity v. Likewize, assume that when $t' \neq 0$ that $t' = (t - vx)\, g(v)$ where $g(v)$ is also a function of the magnitude of the velocity v.

We know that $t' = 0$ when $t = vx$, so $t' = (t - vx)\, g(v)$.
Then because $c = 1$ in all inertial reference frames, when $x = t$, $x' = t'$.
So suppose that $x = t$, it must be true that if $x' = (x - vt)\, f(v)$ and $t' = (t - vx)\, f(v)$, this is only possible if $f(v) = g(v)$. Also we note that for symmetry, it must also be true that $x = (x' + vt')\, f(v)$ and $t = (t' + vx')\, f(v)$.

So substituting into $x = (x' + vt')\, f(v)$ then
$x = (x' + vt')\, f(v) = ([(x - vt)\, f(v)] + v\,[(t - vx)\, f(v)])\, f(v)$.
Hence $x = xf^2(v) - v^2 x f^2(v) = x\left(f^2(v) - v^2 f^2(v)\right) = x\left(1 - v^2\right) f^2(v)$, so if $x = x\left(1 - v^2\right) f^2(v)$, then $f(v) = \frac{1}{\sqrt{1-v^2}}$.
$\frac{1}{\sqrt{1-v^2}}$. is the Lorentz factor.

Thus if $x' = (x - vt)\, f(v)$, then $x' = \frac{(x-vt)}{\sqrt{1-v^2}}$ and similarly,
if $t' = (t - vx)\, f(v)$, then $t' = \frac{(t-vx)}{\sqrt{1-v^2}}$.

These are the Lorentz transformations expressed in matrical form:
$\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1-v^2}} & \frac{-v}{\sqrt{1-v^2}} \\ \frac{-v}{\sqrt{1-v^2}} & \frac{1}{\sqrt{1-v^2}} \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}$, or if v not expressed as a ratio of c,
then $\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} & \frac{-v/c}{\sqrt{1-\frac{v^2}{c^2}}} \\ \frac{-v/c}{\sqrt{1-\frac{v^2}{c^2}}} & \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}$